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The Eulerian Limit for 2D Statistical Hydrodynamics*

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We consider the 2D Navier–Stokes system, perturbed by a white in time random force, proportional to the square root of the viscosity. We prove that under the limit "time to infinity, viscosity to zero" each of its (random) solution converges in distribution to a non-trivial stationary process, formed by solutions of the (free) Euler equation, while the Reynolds number grows to infinity. We study the convergence and the limiting solutions.

KEY WORDS: 2D Euler equation; 2D Navier–Stokes equation; 2D turbulence; invariant measure; random force; Reynolds number; stationary measure; white noise.

1. INTRODUCTION

In this work we study the small-viscosity 2D Navier–Stokes (NS) system, perturbed by a small random force:

$$\begin{gathered} \dot{u}-\nu\,\Delta u+(u\cdot\nabla)\,u+\nabla p=\sqrt{\nu}\,\hat{\eta}(t,x), \qquad 0<\nu\leqslant 1, \\ \operatorname{div} u=0, \qquad u=u(t,x), \qquad p=p(t,x), \qquad x\in T^2=\mathbb{R}^2/(2\pi L\mathbb{Z}^2). \end{gathered} \tag{1.1}$$

It is assumed that $\int u\,dx\equiv\int\hat{\eta}\,dx\equiv 0$, and that $\hat{\eta}(t,x)=\frac{d}{dt}\hat{\zeta}(t,x)$, where the random field $\hat{\zeta}$ is (sufficiently) smooth in x, while as a function of time t it is a Wiener process (see below (2.4)). That is, the force $\hat{\eta}(t,x)$ is white in time and smooth in space.

Equation (1.1), interpreted as a Markov system in the space H of square-integrable divergence-free vector fields $u(x)$, has a stationary

* Dedicated to Giovanni Jona-Lasinio on his 70th birthday.

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measure μ_ν (see ref. 24). Moreover, the equation has a (weak) stationary solution $U_\nu(t)$, $t \in \mathbb{R}$, such that $\mathscr{D}U_\nu(t) \equiv \mu_\nu$. It was proved recently that under certain nondegeneracy assumptions on the force, the stationary measure μ_ν is unique, and distribution $\mathscr{D}(u_\nu(t))$ of any solution u_ν converges to μ_ν as $t \to \infty$. So in the non-degenerate case we have:

$$\operatorname{dist}(\mathscr{D}(u_\nu(t), \mathscr{D}U_\nu(t)) \to 0 \qquad \text{as} \quad t \to \infty. \tag{1.2}$$

Discussion of this result see below in Section 2.

Let $U_\nu(t)$ be any stationary solution of (1.1) (we do not assume that the force $\hat{\eta}$ is non-degenerate. Well known calculations, based on the Ito formula (repeated in Section 2) show that

$$\mathbb{E}\, \|U_\nu\|_1^2 = \tfrac{1}{2} B_0, \qquad \mathbb{E}\, \|U_\nu\|_2^2 = \tfrac{1}{2} L^{-2} B_1, \tag{1.3}$$

where B_0 and B_1 are explicit constants, calculated in terms of the force $\hat{\eta}$, and for $m = 0, 1, 2,...$ we denote by $\|\cdot\|_m$ the norm in the Sobolev space $H^m \subset H$. For $m = 0$ $\|\cdot\|_0$ is the L_2-norm, it will be denoted $|\cdot|$.

The estimates (1.3) imply that the averaged energy[2] $\mathbb{E}\, |U_\nu(t)|^2$ of U_ν is of order L^2 (see below (2.15)). Hence, its Reynolds number is

$$R(U_\nu) = \frac{L(\mathbb{E}\, |U_\nu|^2)^{1/2}}{\nu} \sim \frac{L^2}{\nu}.$$

The fact that the energy of U_ν is of order L^2 while its Reynolds number is of order $L^2 \nu^{-1}$ makes our scaling of the NS system convenient to study the 2D turbulence. We note that the substitution $u(t) = \nu^{1/3} v(\tau)$, $\tau = \nu^{1/3} t$, transforms (1.1) to the equation

$$v_\tau - \delta\, \Delta v + (v \cdot \nabla)\, v + \nabla p' = \frac{d}{d\tau}\, \tilde{\zeta}(\tau, x), \tag{1.4}$$

where $\delta = \nu^{2/3}$ and $\tilde{\zeta}(\tau, x) = \nu^{1/6} \zeta(\nu^{-1/3}\tau, x)$. The random field $\tilde{\zeta}$ is distributed as $\hat{\zeta}$, and stationary solutions for (1.4) have energy of order $L^2\delta^{-1}$ and the Reynolds number of order $L^2\delta^{-3/2}$. So (1.1) is equivalent to the NS system (1.4), perturbed by a white in time random force of order one, but the former scaling is more convenient than the latter.

Our goal in this work is to examine the inviscid limit of the process U_ν, i.e., its limit as $\nu \to 0$. The period L is a parameter of the problem which we keep fixed.

[2] More exactly, the doubled energy per unit area since $|U_\nu|^2 = (2\pi L)^{-2} \int |U_\nu(x)|^2\, dx$.

In Theorem 3.1 and Lemma 3.2 we show that any sequence $\tilde{\nu}_j \to 0$ contains a subsequence $\nu_j \to 0$ such that the process U_{ν_j} converges in distribution to a limiting process $U(\cdot)$. Its energy $|U(t)|^2$ and enstrophy $\|U\|_1^2 = |\mathrm{rot}\, U|^2$ are time-independent random constants:

$$|U(t)|^2 = E, \qquad \|U(t)\|_1^2 = \Omega \qquad \forall t.$$

The expectation of the enstrophy is known, while the expectation of the energy is bounded from below and from above:

$$\mathbb{E}\Omega = \frac{1}{2} B_0, \qquad \frac{L^2 B_0^2}{2B_1} \leqslant \mathbb{E} E \leqslant \frac{L^2 B_0}{2}. \tag{1.5}$$

Moreover, certain exponential moments of E and Ω are finite, and

$$\mathbb{E}\, \|U(t)\|_2^2 \leqslant \tfrac{1}{2} L^{-2} B_1 \qquad \forall t. \tag{1.6}$$

Crucial property of the process U, established in Theorem 3.4, is that almost every of its trajectory satisfies the Euler equation

$$\dot{U} + (U \cdot \nabla)\, U + \nabla p = 0, \qquad \mathrm{div}\, U = 0, \qquad \int U\, dx = 0. \tag{1.7}$$

Note that although U solves the free equation (1.7), relations (1.5) show that it remembers some characteristics of the force $\hat{\eta}$. In particular, U is not identically zero. In Section 3.5 it is proved that if the force $\hat{\eta}(t, x)$ is non-degenerate and stationary in x, then the process U has zero mean-value:

$$\mathbb{E} U(t, x) = 0 \qquad \forall t, x.$$

Due to (1.5) this implies that the process U is "genuinely random."

Due to (1.6), $U(0) \in H^2$ a.s. No existence and uniqueness theorem for the Euler equation with Cauchy data in H^2 is known. Still, as we show in Section 3.4, estimates (1.5) and (1.6) imply that with probability one trajectories of the process U belong to a uniqueness class for the Euler equation, and that they are trajectories of a continuous dynamical system which the equation defines in a certain functional space H^E. Accordingly, the distribution $\mathscr{D}U(t)$ of the process U at any point t is an invariant measure for the Euler equation in the space H^E (see Theorem 3.6).

Jointly with (1.2) the established results imply that in the non-degenerate case the double limit

$$\lim_{\nu_j \to 0}\ \lim_{T \to \infty}\ u_{\nu_j}(T + \cdot) = U(\cdot) \tag{1.8}$$

exists (the convergences are understood as $*$-weak convergences of distributions of random processes $\{u(t),\ t \geqslant 0\}$), and the limiting stationary process U is a random solution of the deterministic Euler equation (1.7). The limits in (1.8) do not commute. Indeed, when $\nu_j \to 0$, $u_{\nu_j}(t)$ converges to a deterministic solution $u_0(t)$ of (1.7). When $t \to \infty$ this solution has no reason to converge to a limit and to remember characteristics of the force $\tilde{\eta}$ (cf. (1.5)).

Our results show that the scaling (1.1) of the 2D NS system may be appropriate for certain form of 2D turbulence since under the limit $\nu \to 0$ stationary solutions of the equation converge (at least, along a subsequence) to a regular limit, while their Reynolds numbers diverge to infinity. In the 3D case the right scaling is different, if we believe that predictions of the Kolmogorov theory apply to space-periodic turbulent flows. Indeed, let u_ν be a stationary (in time) solution of the 3D NS system (1.1) with $\sqrt{\nu}\,\hat{\eta}$ replaced by $\hat{\eta}$.[3] Then, applying the Ito formula, we get that $\nu\,\mathbb{E}\,\|u_\nu\|_1^2 = \frac{1}{2} B_0$ (cf. (1.3)). In particular, $\nu\,\mathbb{E}\,\|u_\nu\|_1^2$ (trivially) converges to a positive finite limit when $\nu \to 0$, as predicted by the Kolmogorov theory. So in the 3D case exactly this scaling (without the factor $\sqrt{\nu}$ in the r.h.s.) should be correct. We also note that our results apparently disagree with the Kraichnan theory of 2D turbulence (ref. 13 and Section 4.2 in ref. 12). Actually, Kraichnan's theory involves an inverse cascade which cannot be present in a bounded system, so it is ruled out by our space-periodic solutions of the white-forced NS system. Indeed, if u_ν is a stationary solution of (1.1), then $\mathbb{E}\,\|u_\nu(t)\|_2^2 = O(1)$ as $\nu \to 0$, see (1.3). In particular $\lim \nu\,\mathbb{E}\,\|u_\nu\|_2^2 = 0$. But it is postulated in the Kraichnan theory that this limit is positive and finite.[4]

Due to (1.8), solutions of the Euler equation approximate the velocity fields of 2D turbulent flows with high Reynolds numbers. This property of the 2D turbulence was predicted by many physicists, cf. refs. 12 and 21. Accordingly, one can try to use the Euler equation to calculate characteristics of solutions u_ν for (1.1) with $\nu \ll 1$. For example, let us take a continuous functional f on H, consider an observable variable $f(u_\nu(t))$ and try to find its average in ensemble. If we can use some ergodic arguments to calculate the averages $\langle f \rangle$ of f along trajectories of the Euler flow, then due to (1.8) we will have $\lim_{\nu_j \to 0} \lim_{t \to \infty} \mathbb{E} f(u_{\nu_j}(t)) = \langle f \rangle$, independently of the sequence ν_j. So $\langle f \rangle$ would approximate asymptotic in time value

[3] We boldly believe that, as in the 2D case, such solutions exist.

[4] Note that for any other scaling of the 2D NS equation (e.g., for (1.4)) the inviscid limit $\lim_{\nu \to 0} u_\nu$ is either zero or infinity.

of $\mathbb{E} f(u_\nu(t))$ for solutions with small viscosity ν. As we show in Proposition 3.7, the quantities $I_g = \int g(\operatorname{rot} U(t, x))\, dx$, where g are bounded continuous functions, are integrals of motion for the limiting processes U. Hence, their trajectories cannot be ergodic in the codimension-two surfaces $H_{a,b} = \{|u| = a, \|u\|_1 = b\}$. So it is unlikely that the "right" average $\langle f \rangle$ can be found, assuming equidistribution of trajectories on the surfaces $H_{a,b}$ (concerning such calculations see ref. 12). On the contrary, it is plausible that the energy $|U|^2$ and the quantities I_g give a complete list of integrals of motion for the process U. If so, then the theory of Robert (e.g., see in ref. 22) can be used to calculate the averages of f along the corresponding surfaces of infinite codimension. Next one can hope to integrate the resulting quantities in the parameter, which parameterises the surfaces (at least, to integrate approximately). This would give us approximate value of the expectations $\mathbb{E} f(u_\nu(t))$ for large time t and small viscosity ν.

The approach to study limiting behaviour of the NS system, suggested in this work, applies to other damped/driven Hamiltonian PDEs with two or more integrals of motion. In particular, to the damped/driven nonlinear Schrödinger equation

$$\dot{u} - \nu\, \Delta u + i(\Delta u - |u|^2 u) = \sqrt{\nu}\, \eta(t, x), \qquad \dim x \leqslant 4.$$

Now the limiting processes U are non-trivial stationary solutions of the nonlinear Schrödinger equation, and the distributions $\mathscr{D}U(0)$ are its invariant measures. Details will be given in a joint paper with Armen Shirikyan which is now under preparation.

Notations. $\mathscr{P}(X)$ is the set of probability Borel measures on a metric space X, provided with the $*$-weak topology. $C_b(X)$ is the space of bounded continuous functions on X. $\mathscr{D}(\xi)$ denotes the distribution of a random variable ξ. By C, C_1, etc. we denote various ν-independent finite constants.

2. PRELIMINARIES

Let H be the Hilbert space of square-integrable divergence-free vector fields on T^2 with zero mean-value, given the scalar product $(u, v) = (2\pi L)^{-2} \int u \cdot v\, dx$ and the norm $|u| = (u, u)^{1/2}$. Let Π be L_2-orthogonal projector to H. Applying Π to (1.1) we get

$$\dot{u}(t) + \nu A u(t) + B(u(t), u(t)) = \nu^{1/2} \eta, \tag{2.1}$$

where $A = -\Pi \Delta$, $B(u, u) = \Pi (u \cdot \nabla)\, u$, and $\eta = \Pi \hat{\eta}$. In (2.1) we view u and η as curves in H. See, e.g., refs. 3 and 9.

For $\nu=0$ Eq. (1.1) becomes the Euler equation (1.7). Application of the projector Π transforms it to Eq. (2.1) with $\nu=0$:

$$\dot{u}(t)+B(u(t), u(t))=0.$$

Let $\{e_s, s\in\mathbb{Z}_0^2 := \mathbb{Z}^2\setminus\{0\}\}$ be the Hilbert basis of H, formed by trigonometric vector fields:

$$\{e_s, e_{-s}\}=\{c_s s^\perp \cos L^{-1}s\cdot x,\ c_s s^\perp \sin L^{-1}s\cdot x\} \qquad \forall s\in\mathbb{Z}_0^2,$$

where $s^\perp=\binom{-s_2}{s_1}$ for any vector $\binom{s_1}{s_2}\in\mathbb{Z}_0^2$, and $c_s=\sqrt{2}\,|s|^{-1}$ is the L_2-normalizing factor. Then

$$Ae_s=\alpha_s e_s, \qquad \alpha_s=L^{-2}\,|s|^2 \qquad \forall s.$$

For any $r\geqslant 0$ we define the Sobolev space H^r as $H^r=\{u\in H \mid \|u\|_r=|A^{r/2}u|<\infty\}$. In particular, $H^0=H$ and $\|\cdot\|_0=|\cdot|$. For $r<0$ we set H^r to be equal to the completion of H in the norm $\|\cdot\|_r$, defined as above.

Since for $u=\sum u_s e_s$ we have $\|u\|_1^2=\sum \alpha_s\,|u_s|^2$ and $\alpha_s\geqslant L^{-2}$ for each s, then

$$\|u\|_1\geqslant L^{-1}\,|u|. \tag{2.2}$$

The norms $\|\cdot\|_j$ satisfy the interpolation inequalities. In particular,

$$\|u\|_1^2\leqslant |u|\cdot\|u\|_2. \tag{2.3}$$

The random force η is assumed to be

$$\eta(t)=\frac{d}{dt}\zeta(t), \qquad \zeta=\sum_{s\in\mathbb{Z}_0^2} b_s\beta_s(t)\,e_s(x), \tag{2.4}$$

where $\{\beta_s\}$ are independent standard Wiener processes, defined for $t\in\mathbb{R}$, and $\{b_s\geqslant 0\}$ are constants. For $j=0, 1,...$ we set

$$B_j=\sum_{s\in\mathbb{Z}_0} |s|^{2j}\,b_s^2\leqslant\infty.$$

Everywhere below we assume that

$$B_0, B_1<\infty. \tag{2.5}$$

Integrating (2.1) from τ to $T>\tau$ we get

$$u(T)+\int_\tau^T (\nu Au+B(u,u))\,dt=u(\tau)+\sqrt{\nu}\,\zeta(T)-\sqrt{\nu}\,\zeta(\tau). \tag{2.6}$$

An a.s. continuous random process $u(t) \in H$, $t \geqslant \tau$, such that its norm $\|u(t)\|_1$ is square-integrable on every finite time-interval is called a *solution* for (2.1), if the relation (2.6) holds a.s. for each $T > \tau$. Here both parts of (2.6) are understood as elements of H^{-1}. It is known (see, e.g., ref. 24) that for any $u_0 \in H$ Eq. (2.1) has a unique solution $u(t)$, $t \geqslant \tau$, equal u_0 for $t = \tau$.

A random process $\tilde{u}(t) \in H_0$, $t \geqslant \tau$, is called a *weak solution* for (2.1) if one can find a process $\tilde{\zeta}(t) \in H$, distributed as $\zeta(t)$, such that $\tilde{u}$ is a solution for (2.1) with ζ replaced by $\tilde{\zeta}$. Its distribution $\mathscr{D}(\tilde{u}(\cdot))$ (i.e., a measure in $C([\tau, \infty); H))$ also is called a week solution, cf. ref. 24, Chapter X. A solution $u(t)$ (weak or strong), defined for all t, is called a *stationary solution* if u is a stationary process in H. For a random variable $v \in H$ with a finite second moment, independent of the process $\eta(t)$, $t \geqslant 0$, we denote by $u(t; v)$, $t \geqslant 0$, a strong solution for (2.1), equal v at $t = 0$ (such a solution exists, see ref. 24). A measure $\mu \in \mathscr{P}(H)$ is called a *stationary measure* for (2.1) if

$$\mathscr{D}(v) = \mu \Rightarrow \mathscr{D}u(t; v) = \mu \qquad \forall t \geqslant 0.$$

If $u(\cdot)$ is a stationary solution, then the distribution $\mathscr{D}u(t)$, $t \in \mathbb{R}$, is a (t-independent) stationary measure in H. Conversely, if μ is a stationary measure, then there exists a (weak) stationary solution u such that $\mathscr{D}u(t) \equiv \mu$ (e.g., see in refs. 8 and 16).

It is known that every Eq. (2.1) (satisfying (2.5)) has a stationary measure μ, see ref. 24. Its uniqueness is established under the additional assumption that

$$b_s \neq 0 \qquad \forall |s| \leqslant N_\nu, \tag{2.7}$$

where N_ν is a sufficiently large number. First this result was proven in ref. 14 for the NS system (as well as for some other nonlinear equations), perturbed by random kick-forces

$$\sum_{k=-\infty}^{\infty} \eta_k(x)\, \delta(t - Tk), \qquad \eta_k(x) = \sum_s b_s \xi_{sk} e_s(x),$$

where $\{\xi_{sk}\}$ are independent bounded random variables with k-independent distributions, satisfying some mild restrictions. The proof is based on the Foias–Prodi reduction which reduces the NS system to a finite-dimensional system with delay, satisfied by Fourier coefficients $u_s(t)$ of solutions $u(t, x)$, with $|s| \leqslant N_\nu$. Next E *et al.*[8] and Bricmont *et al.*[1] applied the Foias–Prodi reduction to prove that the white-forced NS system (2.1), (2.4), (2.7) has a unique stationary measure. These results were later developed in a number of works, including refs. 6, 15, 19, and 20, and ref. 16 (see the review in ref. 17). In particular, it was shown in refs. 1 and 19 (under some additional restrictions) and later in ref. 16 (without the restrictions) that

$$\left| \mathbb{E} f(u(t; v)) - \int f(u)\, \mu\, du \right| \leqslant C_\nu (1 + \mathbb{E}\, |v|^2)\, e^{-\sigma_\nu t}, \qquad \sigma_\nu > 0, \tag{2.8}$$

for any bounded Lipschitz functional f on H such that $|f| \leqslant 1$ and $\mathrm{Lip}(f) \leqslant 1$.

Let us introduce in $\mathscr{P}(H)$ the distance

$$\mathrm{dist}(\mu_1, \mu_2) = \sup(\mu_1 - \mu_2, f),$$

where the supremum is taken over all f as above. It is known that $(\mathscr{P}(H), \mathrm{dist})$ is a complete metric space, and that convergence with respect to this distance is equivalent to the $*$-weak convergence of measures, see ref. 5. We can rewrite (2.8) as

$$\mathrm{dist}(\mathscr{D}u(t; v), \mu) \leqslant C_\nu (1 + \mathbb{E}\, |v|^2)\, e^{-\sigma_\nu t}. \tag{2.9}$$

We note that if

$$b_s \neq 0 \qquad \forall s, \tag{2.10}$$

then the assumption (2.7) holds for each $\nu > 0$, so for any $\nu > 0$ Eq. (2.1) has a unique stationary measure $\mu = \mu_\nu$, satisfying (2.9).

Lemma 2.1. If (2.10) holds, then the measure $\mu_\nu \in \mathscr{P}(H)$ continuously depends on $\nu > 0$.

Proof. Due to (2.9) it is sufficient to check that $\mathscr{D}u_\nu(t) \in \mathscr{P}(H)$ continuously depends on ν for each $t \geqslant 0$, where $u_\nu(\cdot)$ is a solution of (2.1), vanishing at $t = 0$. To verify this property we make in (2.1), (2.4) the substitution $u = \nu U(\tau\nu)$ and get for U the equation

$$U_\tau + AU + B(U, U) = \nu^{-5/2} \frac{\partial}{\partial \tau} \zeta(\nu\tau) =: \frac{\partial}{\partial \tau} \zeta_\nu(\tau).$$

To prove the continuity we may assume that $\nu > \delta > 0$. Then $0 \leqslant \tau \leqslant t/\nu < T = t/\sigma$. It is easy to check that a solution $U \in C([0, T]; H) = \mathscr{H}$ continuously depends on $\zeta_\nu \in C([0, T]; H^1) = \mathscr{H}^1$. Since a.s. $\zeta \in C([0, \infty), H^1)$, then a.s. $\zeta_\nu \in \mathscr{H}^1$ continuously depends on $\nu \in [\delta, 1]$, as well as $U \in \mathscr{H}$. As $u_\nu(t) = \nu U(t\nu^{-1})$, then a.s. $u_\nu(t) \in H$ continuously depends on ν. Hence, $\mathscr{D}u_{\nu'}(t) \to \mathscr{D}u_\nu(t)$ as $\nu' \to \nu$, and the lemma is proven. ▮

Let $u(t)$ be a stationary solution of (2.1) (weak or strong). Due to (2.5) the Ito formula applies to the functional $|u(t)|^2$. Since $(B(u, u), u) = 0$, then taking expectation of the formula we get that

$$\mathbb{E}\, |u(t)|^2 - \mathbb{E}\, |u(0)|^2 = -2\nu \int_0^t \mathbb{E}(Au, u)\, d\tau + \nu \sum_s \int_0^t b_s^2\, |e_s|^2\, d\tau,$$

(see refs. 8, 23, and 24). As u is a stationary process, then this relation implies that

$$\mathbb{E}\, \|u(\tau)\|_1^2 = \tfrac{1}{2}\, B_0 \qquad \forall \tau. \tag{2.11}$$

Since $u = u(x)$ is a divergence-free vector field, then

$$\|u\|_1^2 = (2\pi L)^{-2} \int_{T^2} |\mathrm{rot}\, u|^2\, dx.$$

So the l.h.s. of (2.11) is the averaged enstrophy of u (per unit area).

Similar, applying the Ito formula to the enstrophy functional $f(u) = \|u\|_1^2$ and using that $(B(u, u), Au) = 0$, we obtain

$$\mathbb{E}\, \|u(\tau)\|_2^2 = \tfrac{1}{2}\, L^{-2} B_1 \qquad \forall \tau. \tag{2.12}$$

Two useful inequalities below follow by applying the Ito formula to the functionals $\exp(\gamma\, |u|^2)$ and $\exp(\gamma\, \|u\|_1^2)$, where

$$\gamma = (2L^2 b_{\max}^2)^{-1}, \qquad b_{\max} = \max\{b_s,\, s \in \mathbb{Z}_0^2\}.$$

Namely,

$$\begin{aligned} \mathbb{E} \exp(\gamma\, |u(\tau)|^2) &\leqslant \mathfrak{B}_0 = B_0 \exp\left(\frac{B_0 + 1}{2b_{\max}^2}\right), \\ \mathbb{E} \exp(\gamma\, \|u(\tau)\|_1^2) &\leqslant \mathfrak{B}_1 = L^{-2} B_1 \exp\left(\frac{L^{-2} B_1 + 1}{2b_{\max}^2}\right). \end{aligned} \tag{2.13}$$

See ref. 24, p. 395 and ref. 23. Due to (2.13),

$$\mathbb{E}\, \|u(\tau)\|_1^p \leqslant C_p \gamma^{-p/2} \mathfrak{B}_1 \tag{2.14}$$

for any $p \geqslant 1$.

Equalities (2.11) and (2.12) imply lower and upper bounds for $\mathbb{E}\, |u(\tau)|^2$. Indeed, taking the expectation of (2.3) we find that

$$\mathbb{E}\, \|u(\tau)\|_1^2 \leqslant (\mathbb{E}\, |u(\tau)|^2)^{1/2}\, (\mathbb{E}\, \|u(\tau)\|_2^2)^{1/2}.$$

Now (2.11), (2.12) imply a lower bound for $\mathbb{E}\,|u(\tau)|^2$. Combining it with an upper bound, which follows from (2.2) and (2.11), we get

$$\frac{1}{2}\frac{L^2B_0^2}{B_1} \leqslant \mathbb{E}\,|u(\tau)|^2 \leqslant \frac{L^2B_0}{2}. \tag{2.15}$$

3. THE RESULTS

3.1. Tightness

Let μ_ν be a stationary measure for Eq. (2.1) and $u_\nu(t)$, $t \in \mathbb{R}$, be a corresponding stationary solution (i.e., $\mathscr{D}(u_\nu(t)) \equiv \mu_\nu$). The equality (2.12) and the Prokhorov theorem immediately imply that the set $\{\mu_\nu, \nu \in (0, 1]\}$ is tight in $H^{2-\epsilon}$ for any $\epsilon > 0$. Jointly with Lemma 2.1 this shows that if $b_s \neq 0$ for all s, then the measures $\mu_\nu \in \mathscr{P}(H^{2-\epsilon})$ continuously depend on $\nu \in (0, 1]$.

What is much more important, distributions $m_\nu = \mathscr{D}(u_\nu(\cdot)) \in \mathscr{P}(C(\mathbb{R}; H))$ of the weak stationary solutions u_ν for (2.1) also are tight. Below we prove the tightness for the most important case $\nu \to 0$.

Let us fix any $\varepsilon \in (0, 1)$ and define the following spaces of trajectories:

$$Z = L_{2\,\mathrm{loc}}(\mathbb{R}, H^{2-\varepsilon}) \cap C(\mathbb{R}, H^{-1-\varepsilon}),$$
$$Z_n = L_2([-n, n], H^{2-\varepsilon}) \cap C([-n, n], H^{-1-\varepsilon}), \qquad n \in \mathbb{N}.$$

The norms in the spaces Z_n define a countable system of semi-norms $[\cdot]_n$ in Z, and define there the distance

$$\mathrm{dist}(u, v) = \sum 2^{-n} \frac{[u-v]_n}{[u-v]_n + 1},$$

cf. ref. 24, p. 340. All the spaces Z_n and Z are Polish (i.e., complete and separable).

Theorem 3.1. Any sequence $\{m_{\tilde\nu_j}\}$, $\tilde\nu_j \to 0$, contains a converging subsequence

$$m_{\nu_j} \to m \in \mathscr{P}(Z). \tag{3.1}$$

The theorem is proven in Section 4.1.

The measures m_{ν_j} are stationary since they are distributions of stationary processes, so the limiting measure m is stationary as well. By the

Skorokhod theorem (see ref. 11, Section I.2, we can find random processes $U_{\nu_j}(t)$, $j \geqslant 1$, and $U(t)$, defined on the same probability space, such that

$$\mathscr{D}U_{\nu_j} = m_{\nu_j} \quad \forall j, \qquad \mathscr{D}U = m,$$

and

$$U_{\nu_j} \to U \qquad \text{in } Z \text{ a.s.} \tag{3.2}$$

Below we study properties of the limiting random process U.

3.2. Estimates for U_{ν_j} and U

For any $N \geqslant 1$ we consider the projection

$$P_N\colon H \to H, \qquad \sum_{s \in \mathbb{Z}_0^2} u_s e_s \mapsto \sum_{|s| \leqslant N} u_s e_s,$$

and define the functionals

$$f(u) = e^{\gamma \|u\|_1^2}, \qquad f_N(u) = e^{\gamma \|P_N u\|_1^2 \wedge N}, \qquad N \geqslant 1.$$

Clearly f is continuous on H^1, the functionals f_N are bounded continuous on H^{-2} and

$$0 < f_N(u) \nearrow f(u) \leqslant \infty \qquad \forall u \in H^{-2}. \tag{3.3}$$

Everywhere below in this section

$$\nu \in \{\nu_1, \nu_2, \ldots\},$$

where $\nu_1, \nu_2, \ldots$ is the sequence in (3.1).

Due to (3.2), for each $\tau \in \mathbb{R}$ we have $\mathbb{E} f_N(U_{\nu_j}(\tau)) \to \mathbb{E} f_N(U(\tau))$ as $j \to \infty$. So $\mathbb{E} f_N(U(\tau)) \leqslant \mathfrak{B}_1$ for each N, due to (2.13). Evoking (3.3) and the Levi theorem we get that

$$\mathbb{E} \exp(\gamma \|U(\tau)\|_1^2) \leqslant \mathfrak{B}_1 \qquad \forall \tau. \tag{3.4}$$

Similar,

$$\mathbb{E} \exp(\gamma |U(\tau)|^2) \leqslant \mathfrak{B}_0 \qquad \forall \tau. \tag{3.5}$$

A remarkable property of the process U is that a.s. its energy $|U(t)|^2$ and enstrophy $\|U(t)\|_1^2$ are time-independent random constants:

Lemma 3.2. Almost surely we have

$$|U(t)|^2 = E \quad \text{and} \quad \|U(t)\|_1^2 = \Omega \qquad \text{for a.a.} \quad t \in \mathbb{R}. \tag{3.6}$$

The expectations of the random constants E and Ω satisfy

$$\mathbb{E}\Omega = \frac{1}{2} B_0, \qquad \frac{L^2 B_0^2}{2B_1} \leqslant \mathbb{E} E \leqslant \frac{L^2 B_0}{2}. \tag{3.7}$$

Besides,

$$\mathbb{E}\, \|U(\tau)\|_2^2 \leqslant \tfrac{1}{2} L^2 B_1 \qquad \forall \tau. \tag{3.8}$$

The lemma is proven below in Section 4.2.

3.3. The Equation for *U*

Since U_ν, $\nu \in \{\nu_j\}$, is a weak solution for (2.1), then for any $T > \tau$ it satisfies (2.6), where ζ is replaced by a process $\tilde{\zeta}_\nu$ with the same distribution. We rewrite (2.6) as follows:

$$\begin{aligned} \Psi(U_\nu) &:= U_\nu(T) - U_\nu(\tau) + \int_\tau^T B(U_\nu, U_\nu)\, dt \\ &= -\nu \int_\tau^T AU_\nu(s)\, ds + \sqrt{\nu}(\tilde{\zeta}_\nu(T) - \tilde{\zeta}_\nu(\tau)) =: G_\nu. \end{aligned} \tag{3.9}$$

Lemma 3.3. The nonlinearity B defines continuous quadratic maps $H^2 \to H^1$ and $H^1 \to L_p = L_p(T^2, \mathbb{R}^2) \cap H$, for any $p < 2$.

The proof of the lemma's assertions is obvious since the multiplication of functions defines continuous bi-linear maps $H^2(T^2) \times H^1(T^2) \to H^1(T^2)$ and $H^1(T^2) \times L_2(T^2) \to L_p(T^2)$, and since the projection Π is continuous in the L_p-spaces with $1 < p < \infty$.

Due to the lemma, Ψ defines a continuous map $Z \to H_{-1-\varepsilon}$. So $\Psi(U_{\nu_j}) \to \Psi(U)$ in $H_{-1-\varepsilon}$ as $\nu_j \to 0$, almost surely. From other hand, since

$$\mathbb{E}\, |G_\nu| \leqslant C\sqrt{\nu}$$

due to (2.12) and (2.4), then $G_{\nu_j} \to 0$ in $H_{-1-\varepsilon}$ in probability. Passing in (3.9) to the limit in probability as $\nu_j \to 0$ we get that $\Psi(U) = 0$ a.s. That is, almost surely

$$\dot{U}(t) + B(U(t), U(t)) = 0. \tag{3.10}$$

So, a.s. $U(\cdot) \in Z$ is a weak solution of the Euler equation (3.10), equivalent to (1.7). Due to (3.6), a.s. $\|U(t)\|_1 = \text{const}$. Using (3.8) and the Fubini theorem we get that

$$U(\cdot) \in L_{2\,\text{loc}}(\mathbb{R}; H^2) \quad \text{a.s.} \tag{3.11}$$

Therefore due to Lemma 3.3 and (3.10),

$$\dot{U} \in L_{1\,\text{loc}}(\mathbb{R}; H^1) \cap L_\infty(\mathbb{R}; L_p) \quad \text{a.s.,} \qquad \forall p < 2. \tag{3.12}$$

We have proven the following result:

Theorem 3.4. Any limiting measure $m \in \mathscr{P}(Z)$ as in (3.1) is a weak stationary solution for the Euler equation (3.10) in the following sense: There exists a stationary random process $U(t) \in H$, distributed as m, which a.s. satisfies (3.10). Moreover,

(1) energy $|U(t)|^2$ of this process a.s. is a time-independent random constant E such that

$$\frac{L^2 B_0^2}{2B_1} \leqslant \mathbb{E}E \leqslant \frac{L^2 B_0}{2},$$

(2) its enstrophy $\|U(t)\|_1^2$ a.s. is a time-independent random constant Ω such that $\mathbb{E}\Omega = \frac{1}{2} B_0$;

(3) the process satisfies the estimate (3.8), the exponential estimates (3.4) and (3.5), and its time-derivative satisfies (3.12).

3.4. A Phase-Space for the Euler Equation

By (3.11) and (3.12), with probability one trajectories of the process U belong to the space

$$\mathscr{K} = \{u \in L_{2\,\text{loc}}(\mathbb{R}; H^2) \mid \dot{u} \in L_{1\,\text{loc}}(\mathbb{R}; H^1)\}.$$

Since $\mathscr{K} \subset C(\mathbb{R}; H^1)$, then due to the lemma below $\mathscr{K}$ is an uniqueness class for the Euler equation.

Lemma 3.5. Let $u(t)$ and $v(t)$, $t \in [t_1, t_2]$, be two solutions of the Euler equation (3.10) such that

$$\|u(t)\|_1 + \|v(t)\|_1 \leqslant C_1 \qquad \forall t, \tag{3.13}$$

$$\int_{t_1}^{t_2} \|u(t)\|_2^2 \, dt \leqslant C_2, \tag{3.14}$$

and u equals v at some point $t_3 \in [t_1, t_2]$. Then $u(t) \equiv v(t)$.

The lemma follows from more general results, proved in ref. 25 (see there Corollary 7.2). In Section 4.3 we present its direct proof, based on the classical arguments due to Yudovitch[(26)] (also see in ref. 4).

Let us denote by $\mathscr{K}^E$ the set of trajectories in $\mathscr{K}$ which satisfy the Euler equation. Clearly $\mathscr{K}^E$ is a closed subset of $\mathscr{K}$, invariant under the time-translations $u(\cdot) \to u(T + \cdot)$, $T \in \mathbb{R}$. Due to Lemma 3.5 the map

$$\pi \colon \mathscr{K} \to H^1, \qquad u \mapsto u(0),$$

restricted to $\mathscr{K}^E$, defines a continuous embedding $\mathscr{K}^E \to H^1$. We denote by H^E its image $\pi(\mathscr{K}^E)$, and provide H^E with the topology, induced from $\mathscr{K}^E$.

Due to ref. 2, for any $u_0 \in H^3$ the Euler equation has a unique solution $u \in \mathscr{K} \cap L_{\infty\,\mathrm{loc}}(\mathbb{R}; H^3)$, equal u_0 at $t = 0$. This solution, as a point in $\mathscr{K}$, continuously depends on u_0. So

$$H^3 \subset H^E \subset H^1, \tag{3.15}$$

and the inclusions are continuous. We point out that we do not know if the (topological) space H^E is linear, or not.

For any $u_0 \in H^E$, $u = \pi^{-1}(u_0)$ is the unique solution of the Euler equation in then space $\mathscr{K}$, equal u_0 at $t = 0$. Since $u(t) = \pi(u(t + \cdot))$ and the time-translation maps are continuous in $\mathscr{K}$, then the Euler equation defines a group of continuous automorphisms $S_t \colon H^E \to H^E$, $t \in \mathbb{R}$. As $S_t \circ U(\tau) = U(t + \tau)$ and the process U is stationary, then we have:

Theorem 3.6. The distribution $\mathscr{D} = \mathscr{D}\, U(0)$ of the process U as in Theorem 3.4 is an invariant Borel measure for the flow which the Euler equation defines in H^E. The space H^E satisfies (3.15), and $\mathscr{D}(H^E \cap H^2) = 1$.

Since vector fields from H^2 have the modulus of continuity[5] $g(r) = r \max(1, \log r^{-1})$, then any $u(t, x) \in \mathscr{K}$ has the modulus of continuity in x,

[5] Indeed, by the classical embedding theorem, spaces H^r with non-integer $r > 1$ are embedded in Hölder spaces C^{r-1}. So by the interpolation, the space H^2 is embedded in the interpolating Hölder space C^1_* (the Sigmund space). The latter is formed by functions, which have the modulus of continuity $g(r)$, see ref. 4, p. 31.

equal to $\phi(t)\, g(r)$, where $\phi \in L_{2\,\mathrm{loc}}$. Therefore by the Osgood criterion,[10] the differential equation

$$\dot{x}(t) = u(t, x), \qquad x(t_1) = y,$$

has a unique solution. So the flow-maps $S_{t_1}^{t_2} : T^2 \to T^2$, $y \mapsto x(t_2)$, are well defined and continuous (see ref. 4). Accordingly, due to the classical arguments, for any $u \in \mathscr{K}^E$ we have $\omega(t, x) \equiv \omega_0(S_t^0(x))$, where $\omega(t, x) = \mathrm{rot}\, u(t, x)$ (and $\omega_0 = \mathrm{rot}\, u_0$).

Approximating $u(t, x)$ by smooth divergence-free vector fields $u_\epsilon(t, x)$ (say, obtained as convolutions with scalar mollifiers), we approximate any continuous flow-map $S = S_{t_1}^{t_2}$ by measure-preserving smooth maps $T^2 \to T^2$, converging to S at each point, see ref. 10, Theorem 2.1, Section V. Passing to the limit as $\epsilon \to 0$, we find that $\int_{T^2} f(S(x))\, dx = \int_{T^2} f(x)\, dx$ for each continuous function f. Hence, the flow-maps preserve the Lebesgue measure, and

$$\int_{T^2} g(\omega(t, x))\, dx = \int_{T^2} g(\omega_0 \circ S_t^0)(x)\, dx = \int_{T^2} g(\omega_0(x))\, dx,$$

for any t and any $g \in C_b(\mathbb{R})$. That is,

$$\int_{T^2} g(\mathrm{rot}\, U(t, x))\, dx = \mathrm{const} \qquad \forall t \in \mathbb{R}, \tag{3.16}$$

for each $U \in \mathscr{K}^E$. In particular, we have the following result:

Proposition 3.7. For a.a. trajectory of the process U and any bounded continuous function g, the integral in (3.16) is a time-independent random constant.

3.5. Homogeneous Solution

Let us assume that

$$b_s = b_{-s} \neq 0 \qquad \forall s \in \mathbb{Z}_0^2. \tag{3.17}$$

Then the random field $\zeta(t, x)$ is homogeneous in x, i.e., translations of x do not change its distribution (see ref. 17), and (2.1) has a unique stationary measure μ_ν. As a consequence, the measure μ_ν and a corresponding stationary solution $U_\nu(t, x)$ both are homogeneous in x, see ref. 17. Since U_ν

a.s. converges to the random field $U(t) = U(t, x)$ as in Theorem 3.4, then U is homogeneous (i.e., stationary in x).

That is, under the assumption (3.17) the limiting measures m as in (3.1) correspond to random solutions of (1.7) which are stationary both in time and space. In particular, since $\int U\,dx \equiv 0$, then in this case we have

$$\mathbb{E}\, U(t, x) \equiv 0$$

(i.e., $\mathbb{E}U(t) = 0 \in H^{-1-\varepsilon}$ for all t).

3.6. One Degenerate Example

Let us consider a force η of the form (2.4), such that $b_s = 0$ if $|s| > 1$. Then $B_0 = B_1$, so by Theorem 3.4 we have

$$\mathbb{E} E = \frac{L^2 B_0}{2} = L^2 \mathbb{E} \Omega. \tag{3.18}$$

Let us denote $s_1 = (1, 0)^t$ and $s_2 = (0, 1)^t$. Due to (2.2), $|u|^2 \leqslant L^2 \|u\|_1^2$, and the equality holds if and only if $u \in M^4$, where

$$\begin{aligned} M^4 &= \operatorname{span}\{e_{\pm s_1}, e_{\pm s_2}\} \\ &= \left\{ C_1 s_1 \sin\left(a_1 + \frac{x_2}{L}\right) + C_2 s_2 \sin\left(a_2 + \frac{x_1}{L}\right) \,\middle|\, C_1, C_2 \in \mathbb{R},\, a_1, a_2 \in \mathbb{R}/2\pi\mathbb{Z} \right\}. \end{aligned}$$

So (3.18) implies that a limiting solution $U(t)$ as in Theorem 3.4 belongs to M^4 for all t. For any $u \in M^4$ we have

$$\begin{aligned} (u \cdot \nabla)\, u &= L^{-1} C_1 C_2 \left(s_1 \cos\left(a_1 + \frac{x_2}{L}\right) \sin\left(a_2 + \frac{x_1}{L}\right) \right. \\ &\quad \left. + s_2 \sin\left(a_1 + \frac{x_2}{L}\right) \cos\left(a_2 + \frac{x_1}{L}\right) \right) \\ &= -C_1 C_2 \nabla \left(\cos\left(a_1 + \frac{x_2}{L}\right) \cos\left(a_2 + \frac{x_1}{L}\right) \right), \end{aligned} \tag{3.19}$$

so $B(u, u) = 0$. That is, $\dot{U} \equiv 0$ and U is a constant process

$$U(t) = C_1 s_1 \sin\left(a_1 + \frac{x_2}{L}\right) + C_2 s_2 \sin\left(a_2 + \frac{x_1}{L}\right) \qquad \forall t, \tag{3.20}$$

where C_1, C_2 and a_1, a_2 are random variables.

The processes U above are very degenerate because due to (3.19), the 4-dimensional space M^4 is invariant for Eq. (2.1) with $\eta = 0$. The space M^4 is a maximal one with this property since if M is a linear subspace of H, invariant for (2.1) with $\eta = 0$, and $M^4 \subset M$, $M^4 \neq M$, then M is dense in H, see ref. 7. Accordingly, if $b_s \neq 0$ for $|s| \leqslant 2$, then the corresponding processes U have no chances to be as degenerate as (3.20).

4. PROOFS

4.1. Proof of Theorem 3.1

Our arguments in this section closely follow ref. 24, Chapter X.

First we note that to prove the theorem it is sufficient to check that for any $n \geqslant 1$ the sequence of measures $\{m^n_{\tilde{v}_j}, j = 1, 2,...\}$ is tight in Z^n, where

$$m^n_{\tilde{v}_j} = \mathscr{D}(u_{\tilde{v}_j}|_{[-n,n]}) \in \mathscr{P}(C([-n, n]; H)).$$

Indeed, we need to verify that the sequence $\{m_{\tilde{v}_j}\}$ contains a subsequence $\{m_{v_j}\}$ such that

$$(m_{v_j}, f) \to (m, f), \qquad m \in \mathscr{P}(Z), \tag{4.1}$$

for any $f \in C_b(Z)$. It is known that it suffices to check (4.1) for bounded Lipschitz functionals f_L (see ref. 5, Theorem 11.3.3). It is easy to approximate any such f_L by functionals $f^n_L = g^n_L \circ \pi_n$, where π_n is the natural projection $Z \to Z_n$, and g^n_L is a bounded Lipschitz functional on Z_n. Since $(m_v, g^n_L \circ \pi_n) = (m^n_v, g^n_L)$, then (4.1) holds if

$$m^n_{v_j} \to m^n \qquad \forall n, \tag{4.2}$$

where $m^n = (\pi_n)_* m$. If we know that the sequence $\{m^n_{\tilde{v}_j}\}$ is tight in Z_n for each n, then we can use the diagonal process to construct a subsequence $\{v_j\}$ such that (4.2) holds for every n. The measures $m^n \in \mathscr{P}(Z_n)$ form a compatible family and define a measure $m \in \mathscr{P}(Z)$ that satisfies (4.1).

So we need to check that the sequence of measures $\{\mu^n_{\tilde{v}_j}\}$ is tight for any fixed n. Let u_v be a stationary solution for (2.1) such that $\mathscr{D}(u_v) = m_v$. We abbreviate $u = u_v$ and take any $t_1, t_2 \in [-n, n]$, satisfying $t_1 \leqslant t_2$ and $|t_1 - t_2| \leqslant 1$. We have

$$u(t_2) - u(t_1) = -v \int_{t_1}^{t_2} Au(s)\, ds - \int_{t_1}^{t_2} B(u(s), u(s))\, ds + \sqrt{v}(\zeta(t_2) - \zeta(t_1)) \tag{4.3}$$

(strictly speaking, since u is a weak solution, then ζ should be replaced by a suitable process $\tilde{\zeta}$, distributed as ζ. This inaccuracy makes no difference for what follows).

The H-norm of the first integral in the r.h.s. of (4.3) is bounded by

$$\nu\left(\int_{t_1}^{t_2}\|u\|_2\,ds\right)\leqslant \nu\,|t_2-t_1|^{\frac{1}{2}}\left(\int_{t_1}^{t_2}\|u\|_2^2\,ds\right)^{\frac{1}{2}}\leqslant \nu\,|t_2-t_1|^{\frac{1}{2}}\left(\int_{-n}^{n}\|u\|_2^2\,ds\right)^{\frac{1}{2}}.$$

Due to (2.12),

$$\mathbb{E}\left(\int_{-n}^{n}\|u\|_2^2\right)^{\frac{1}{2}}\leqslant (nL^{-2}B_1)^{\frac{1}{2}}.$$

Since $\|B(u,u)\|_{-1}\leqslant C\,\|u\|_1^2$, then H^{-1}-norm of the second integral is bounded by

$$|t_2-t_1|\sup_{[t_1,t_2]}\|u(t)\|_1^2\leqslant |t_2-t_1|\sum_{m=-n}^{n-1}\sup_{[m,m+1]}\|u(t)\|_1^2.$$

An application of the Ito lemma to $\|u(t)\|_1^2$ (cf. (4.6) in the next subsection) bounds the expectation of oscillation of $\|u(t)\|_1^2$ on any segment $[m,m+1]$ by $C\sqrt{\nu}$. So due to (2.11),

$$\mathbb{E}\sum_{m=-n}^{n-1}\sup_{[m,m+1]}\|u(t)\|_1^2\leqslant C_1 n.$$

Finally, due to basic properties of the Brownian motion, the H-norm of the third term in the right-hand side of (4.3) is bounded by

$$\sqrt{\nu}\,|t_2-t_1|^{\frac{1}{3}}\,\xi,$$

where $\xi\geqslant 0$ and $\mathbb{E}\xi\leqslant C_2(n)$.

Let us consider the Hölder space $C^{1/3}([-n,n],H^{-1})=:\mathscr{C}_n^{1/3}$. Due to (2.11) and the estimates for the norms of the three terms in the r.h.s. of (4.3),

$$\mathbb{E}\,|u_\nu|_{\mathscr{C}_n^{1/3}}\leqslant C(n). \tag{4.4}$$

Let Y_n be the space $Y_n=\mathscr{C}_n^{1/3}\cap L_2([-n,n],H^2)$. Since

$$\mathbb{E}\,|u_\nu|^2_{L_2([-n,n],H^2)}=nL^{-2}B_1$$

by (2.12), then using (4.4) we have

$$\mathbb{E}\,|u_\nu|_{Y_n} \leqslant C'(n) \quad \forall \nu.$$

As the space Y_n is compactly embedded in the separable space Z_n (ref. 24, Lemma X.3.1), then the family of measures $\mu_\nu^n = \mathscr{D} u_\nu|_{[-n,n]}$ is tight in Z_n by the Prokhorov theorem. Theorem 3.1 is proven. ∎

4.2. Proof of Lemma 3.2

Let us apply the Ito formula to the functional $\xi_\nu(t) = \|U_\nu(t)\|_1^2$. For any $t_1 \leqslant t_2$ we have:

$$\xi_\nu(t_2) - \xi_\nu(t_1) = -2\nu \int_{t_1}^{t_2} \|U_\nu(t)\|_2^2\, dt + \nu \int_{t_1}^{t_2} \sum_s b_s^2 (e_s, Ae_s)\, dt$$

$$+2\sqrt{\nu} \sum_s b_s \int_{t_1}^{t_2} (U_\nu(t), Ae_s)\, d\beta_s(t)$$

(cf. the remark after the formula (4.3)). Noting that the second term in the r.h.s. equals $\nu L^{-2} B_1 (t_2 - t_1)$, we see that

$$\operatorname{osc} \xi_\nu|_{[0,1]} := \sup_{0 \leqslant t_1 \leqslant t_2 \leqslant 1} |\xi_\nu(t_1) - \xi_\nu(t_2)|$$

$$\leqslant \nu L^{-2} B_1 + 2\nu \int_0^1 \|U_\nu(t)\|_2^2\, dt$$

$$+4\sqrt{\nu} \sup_{0 \leqslant \tau \leqslant 1} \left| \int_0^\tau \left(AU_\nu(t), \sum b_s e_s\, d\beta_s(t) \right) \right|. \tag{4.5}$$

Writing $U_\nu(t)$ as $\sum U_{\nu s}(t)\, e_s$ and applying the Burkholder–Devis–Gundi inequality, we see that the expectation of the third term in the r.h.s. of (4.5) is bounded by

$$4\sqrt{\nu}\, \mathbb{E} \left(\sum_{s \in Z_0^2} \int_0^1 \alpha_s^2 b_s^2 U_{\nu s}(t)^2\, dt \right)^{1/2} \leqslant 4\sqrt{\nu}\, b_{\max} \left(\mathbb{E} \int_0^1 \|U_\nu(t)\|_2^2\, dt \right)^{1/2}$$

$$= 4\sqrt{\nu}\, b_{\max} L^{-1} \sqrt{\frac{B_1}{2}},$$

where to get the last equality we used (2.12). Using (2.12) once again to estimate the expectation of the second term in the r.h.s. of (4.5) we get that

$$\mathbb{E}(\operatorname{osc} \xi_\nu|_{[0,1]}) \leqslant C\sqrt{\nu}. \tag{4.6}$$

Let us denote by $\mathscr{E}$ and g the following functionals on Z:

$$\mathscr{E}(u) = \int_0^1 \|u(t)\|_1^2 \, dt, \qquad g(u) = \int_0^1 \big| \|u(t)\|_1^2 - \mathscr{E}(u) \big| \, dt.$$

By $\mathscr{E}_N$ and g_N, $N \in \mathbb{N}$, we denote their regularized versions:

$$g_N(u) = g(u) \wedge N, \qquad \mathscr{E}_N(u) = \mathscr{E}(u) \wedge N.$$

All these functionals are continuous non-negative, and

$$g_N \nearrow g, \quad \mathscr{E}_N \nearrow \mathscr{E} \qquad \text{as} \quad N \to \infty.$$

Due to (2.11),

$$\mathbb{E}\mathscr{E}(U_\nu) = \int_0^1 \mathbb{E}\, \|U_\nu(t)\|_1^2 \, dt = \tfrac{1}{2} B_0. \tag{4.7}$$

Besides, due to (2.14) we have

$$\mathbb{E}\, \mathscr{E}(U_\nu)^2 = \mathbb{E}\left(\int_0^1 \|U_\nu\|_1^2 \, dt \right)^2 \leqslant \mathbb{E} \int_0^1 \|U_\nu\|_1^4 \, dt \leqslant C'.$$

Therefore,

$$\mathbb{E}\mathscr{E}(U_\nu)\, I_{\mathscr{E}(U_\nu) > M} \leqslant \frac{1}{M}\, \mathbb{E}\mathscr{E}(U_\nu)^2 \leqslant M^{-1} C'.$$

So the system of random variables $\{\mathscr{E}(U_\nu)\}$ is uniformly integrable, and (3.2) implies that

$$\mathbb{E}\mathscr{E}(U) = \lim \mathbb{E}\mathscr{E}(U_\nu) = \tfrac{1}{2} B_0,$$

see ref. 5, Theorem 10.3.6. Since the process $U(t)$ is stationary, then

$$\mathbb{E}\mathscr{E}(U) = \mathbb{E} \int_0^1 \|U(t)\|_1^2 \, dt = \mathbb{E}\, \|U(t)\|_1^2.$$

That is,

$$\mathbb{E}\, \|U(t)\|_1^2 = \tfrac{1}{2} B_0 \qquad \forall t \in [0, 1]. \tag{4.8}$$

Now we return to estimate (4.6). As $|\xi_\nu(t) - \int_0^1 \xi_\nu(\tau)\, d\tau| \leqslant \operatorname{osc} \xi_\nu|_{[0,1]}$ for $0 \leqslant t \leqslant 1$, then due to (4.6) we have

$$0 \leqslant \mathbb{E} g_N(U_\nu) \leqslant \mathbb{E} g(U_\nu) \leqslant C\sqrt{\nu}.$$

Passing to the limit as $\nu_j \to 0$ using (3.2), we get that $\mathbb{E} g_N(U) = 0$ for each N. That is, $g(U) = 0$ a.s., and $\|U(t)\|_1^2 = \mathscr{E}(U)$ a.e. in $[0, 1]$. Repeating these arguments for all half-integer time-intervals $[\frac{n}{2}, \frac{n}{2}+1]$, $n \in \mathbb{Z}$, we get that a.s.

$$\|U(t)\|_1^2 = \Omega \qquad \text{for a.a.} \quad t \in \mathbb{R}. \tag{4.9}$$

Here Ω is a random constant which can be defined, say, as

$$\Omega = \mathscr{E}(U) = \int_0^1 \|U(t)\|_1^2\, dt.$$

Using (4.8) we see that expectation of Ω satisfies the first relation in (3.7).

Applying the Ito formula to the functional $|U_\nu|^2$, we get a bound for oscillation of $|U_\nu(t)|^2$, analogous to (4.5). Next we repeat derivation of (4.9) to find that the energy of the process $U(t)$ a.s. is a time-independent random constant E.

Passing to the limit in (2.12) using (3.2) and the Levi theorem we get (3.8) (cf. the derivation of (3.4)). Repeating the arguments that proved (2.15) and using (4.8) and (3.8), we find that

$$\frac{L^2 B_0^2}{2B_1} \leqslant \mathbb{E} E \leqslant \frac{L^2 B_0}{2}.$$

Now the lemma is proven. ∎

4.3. Proof of Lemma 3.5

To simplify notations we assume that $t_1 = t_3 = 0$ and $t_2 = T > 0$. Below the constants C, C_1, etc. are independent of the parameter $p > 2$.

Let us denote $b(u_1, u_2, u_3) = (B(u_1, u_2), u_3)$ and $w = u - v$. Subtracting from the equation for u the equation for v, multiplying the result in H by w and using that $b(v, w, w) = 0$ we get:

$$\frac{1}{2}\frac{d}{dt}|w|^2 + b(w, u, w) = 0. \tag{4.10}$$

Abbreviating $b(w, u, w) = b$ and applying the Hölder inequality, we get

$$|b| \leqslant C\, |\nabla u|_p\, |w|_{2q}^2, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 2 \tag{4.11}$$

(here and below $|\cdot|_r$ stands for the L_r-norm). It is known that the Sobolev embedding theorem for the space H^1 can be specified as follows:

$$|V|_p \leqslant C\sqrt{p}\, \|V\|_1 \qquad \forall V \in H^1, \tag{4.12}$$

where $C = C(L)$, see ref. 18, Theorem 8.5.[6] Therefore,

$$|\nabla u|_p \leqslant C\sqrt{p}\, \|u\|_2.$$

To estimate $|w|_{2q}$ we write

$$\int |w|^{2q}\, dx = \int |w|^a\, |w|^{2q-a}\, dx \leqslant |w|_2^a\, |w|_{p(2q-a)}^{2q-a}, \qquad a = \frac{2}{q},$$

where we used the Hölder inequality with $P = 2/a = q$, $Q = p$. Since $\|w\|_1 \leqslant \|u\|_1 + \|v\|_1$, then due to (3.13), (4.12) we have

$$|w|_{2q}^2 \leqslant C\, |w|_2^{2q^{-2}} (C_1^2 p(2q-a))^{\frac{2q-a}{2q}}.$$

Noting that $0 < \frac{2q-a}{2q} = \frac{q+1}{pq} \leqslant \frac{2}{p}$, we obtain

$$|w|_{2q}^2 \leqslant C'\, |w|_2^{2q^{-2}}\, p^{2/p} \leqslant C\, |w|_2^{2q^{-2}}.$$

So

$$|b| \leqslant C\sqrt{p}\, \|u(t)\|_2\, |w|_2^{2q^{-2}}. \tag{4.13}$$

Let us denote $g = |w|_2^2$. Relations (4.10) and (4.13) imply the differential inequality

$$\dot{g}(t) \leqslant C\sqrt{p}\, g(t)^{q^{-2}}\, \|u(t)\|_2.$$

[6] In the statement of the theorem in ref. 18 the value of the constant that bounds the norm of the Sobolev embedding is given incorrectly, but right value of the constant immediately follows from the proof (which is correct).

As $g(0)=0$ and $1-q^{-2}=(q+1)/pq$, then applying the Gronwall lemma we get an upper bound for g:

$$g(t) \leqslant \left(C(1-q^{-2})\sqrt{p}\int_0^t \|u(s)\|_2\, ds \right)^{\frac{pq}{q+1}}$$
$$\leqslant \left(C_1 p^{-1/2}\sqrt{t}\left(\int_0^t \|u(s)\|_2^2\, ds\right)^{1/2} \right)^{\frac{pq}{q+1}}.$$

Using (3.14) and passing to the limit as $p \to \infty$, we see that $g(t)=0$ for all $0 \leqslant t \leqslant T$, as stated.

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